Scatters, unavoidable shapes, and crystallization

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Abstract

We study (n, k)-scatters, which are regular n-gon tiles arranged so that each tile shares edges with at least k others. To measure how much freedom there is in arranging scatters, we ask which shapes are unavoidable. It turns out that for a few choices of (n, k) there are infinite unavoidable shapes, but otherwise they are finite. We discuss the infinite case as an analogue of crystallization.

The main result here is that besides the trivial situations when there's a unique scatter, there are only four instances of this. Scatters crystallize nontrivially just when $(n, k) = (5, 3)$, $(7, 3)$, $(10, 4)$, or $(14, 4)$.

1 Introduction

We use tiles to study a simple combinatorial analogue for crystallization. The game we're playing is to arrange regular n-gon tiles in the plane according only to the rule that each tile has at least k neighbors. Any such arrangement will be called an (n, k)-scatter. We emphasize that the number of neighbors doesn't have to be on the nose but is only bounded below. If there's some infinite shape that always appears no matter how we play the game, we say (n, k)-scatters crystallize.

For an example of why we choose this terminology, crystallization in nature often occurs in media of high saturation or density. This is the case here in the sense that crystallized scatters are always as dense as possible (as measured by k). It also seems likely that crystallization implies zero entropy from the point of view of symbolic dynamics. With additional assumptions we prove this after establishing the main result. Finally, in the last section we give an example of crystallized scatters with 5-fold forbidden symmetry. The physics analogy at least conveys much of our intuition for the subject, although we don't mean to suggest that this is a meaningful model of how crystals form in the physical world.

2 Definitions

A *tile* in this article is a regular n-gon with unit length sides. A *shape* is a finite or infinite set of tiles $t_0, t_1, \ldots$, such that for every $i > 1$, t_i has a common edge with at least one

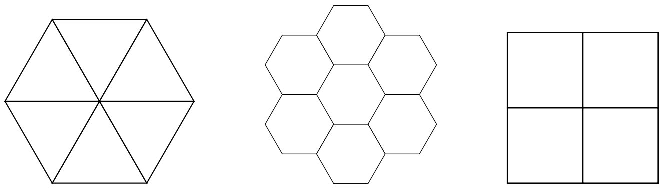

Figure 1: The smallest $(3, 2)$-, $(6, 3)$-, and $(4, 2)$-scatters.

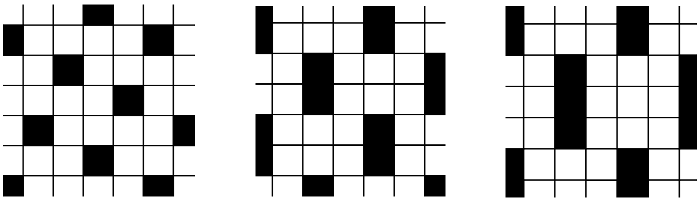

Figure 2: A few of many possible periodic (4,3)-scatters.

t_j with $j < i$, and with tiles only intersecting along their boundaries. We consider two shapes indistinguishable if they're congruent, so the order itself isn't important but just that the tiles are connected in this way.

An (n, k)-*scatter* is a shape satisfying the additional condition that each tile shares edges with at least k others. (We'll use n and k consistently throughout the paper.) A few examples are illustrated in Figures 1 and 2.

A shape is (n, k)-*unavoidable* if a congruent copy appears at least once in every (n, k)-scatter. If there exists any infinite (n, k)-unavoidable shape, we say (n, k)-scatters *crystallize*.

3 Results

We ask first when scatters exist and fortunately there's a simple answer. It's easy to verify that $(n, 3)$-scatters exist for all n and that (n, k)-scatters don't ever exist for $k > 6$. It's also easy to check that $(n, 6)$-scatters exist if and only if $6|n$, in which case there's a unique infinite scatter. (Similarly, there's a unique scatter when $(n, k) = (3, 3)$, $(4, 4)$, or $(8, 4)$. See Figure 3.) So we need only check existence of (n, k)-scatters for $k = 4$ and $k = 5$.

Lemma 3.1. *$(n, 4)$-scatters exist if and only if $2|n$. $(n, 5)$-scatters exist if and only if $6|n$.*

After existence we can turn to our main focus, crystallization. We'll work first to limit the possibilities for when it might occur. The next lemma should be consistent with the reader's intuition - the only chance for things crystallizing is when they're as dense as possible.

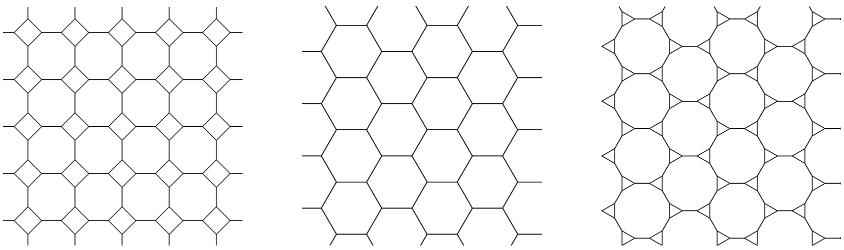

Figure 3: The unique $(8,4)$-,$(6,6)$- and $(12,6)$-scatters.

Lemma 3.2. *If $(n,k+1)$-scatters exist, then (n,k)-scatters don't crystallize.*

Then we need only construct a few explicit examples of each type to rule out crystallizing for almost all other values of n.

Lemma 3.3. *$(n,3)$-scatters don't crystallize for $n \geq 9$, $(n,4)$-scatters don't crystallize for $n \geq 16$.*

Finally, we exhibit unavoidable shapes for each of the choices of parameter we haven't already eliminated and establish the main result.

Theorem 3.4. *Aside from the trivial cases when there's a unique scatter, there are four more which crystallize, when $(n,k) = (5,3)$, $(7,3)$, $(10,4)$, and $(14,4)$.*

In the next three sections we prove these results. Then we briefly discuss entropy, and finally we close with open questions.

4 Existence and nonexistence of $(n,4)$- and $(n,5)$-scatters

Existence is straightforward since we need only construct examples. Whenever n is even, $(n,4)$-scatters exist by the kind of constructions in Figure 5. When $6|n$, $(n,6)$-scatters exist, which are also $(n,5)$-scatters. So all we have to check is that otherwise $(n,4)$-scatters and $(n,5)$-scatters don't exist.

It's convenient here to use the language of graph theory. For any (n,k)-scatter X, define the dual graph of X, G_X, to be the graph whose vertices are tiles in X with two vertices adjacent if the corresponding tiles share an edge in X. So k is a lower bound on minimum vertex degree in G_X.

Recall that the girth of a graph is the length of its shortest cycle. We claim if n is odd then for every (n,k)-scatter X, G_X has girth at least 6. First observe that G_X is bipartite, since two adjacent vertices correspond to tiles where one is rotated π with respect to the other. This rules out odd cycles. But a 4-cycle in the dual graph is geometrically a parallelogram. The interior angles of the parallelogram are $2\pi k/n$ and $2\pi k'/n$ with k and k' integers, and $2\pi k/n + 2\pi k'/n = \pi$. So $n = 2(k+k')$ and n is even. So if n is odd, there's no 4-cycle either and the girth is at least 6.

The dual graph is also an embedded planar graph so it makes sense to talk about faces. We can compute bounds on the average interior angle of a face in two different ways. By the lower bound on girth, the average angle is asymptotically at least $2\pi/3$. But by the lower bound on vertex degree the average angle is asymptotically at most $\pi/2$, a contradiction.

The situation for $(n, 5)$-scatters is similar. If X is an $(n, 5)$-scatter then 3-cycles in G_X are equilateral triangles so if G_X has any triangles then $6|n$. Otherwise the girth is at least 4. So the girth gives an asymptotic lower bound on average face angle of $\pi/2$ and degree gives an asymptotic upper bound of $2\pi/5$.

How is this kind of asymptotic argument justified? Our tilings are nice in the sense that there aren't any singular points where tiles accumulate or any other bad behavior. More precisely, our tilings are *normal* in the sense of Chapter 3 of [Grun] since the tiles are topological disks, the intersection of every two is connected, and the tiles are uniformly bounded in size.

5 Non-crystallized scatters

We prove first Lemma 3.2. If X is any $(n, k+1)$-scatter we can carefully delete tiles from X and still know that what we're left with an (n, k)-scatter. In particular, if S is any set of tiles in X such that no pair of elements in S has a common neighbor in X, then $X - S$ is an (n, k)-scatter.

Suppose by way of contradiction that U is an infinite (n, k)-unavoidable shape. How many distinct congruent copies of U appear in X? At most countably many, so let $T = \{t_1, t_2, \ldots\}$ be an ordered list of all of them. Let s_1 be any tile in t_1, and for $i = 2, 3, \ldots$ let s_i be any tile in t_i such that s_i doesn't have any neighbors in common with s_j for any j with $j < i$. Such a choice always exists since t_i is by assumption unbounded. Write $S = \{s_1, s_2, \ldots\}$. Then $X - S$ is an (n, k)-scatter in which no copy of U appears, a contradiction.

To prove Lemma 3.3 we need only construct enough scatters, and it turns out that a few periodic examples are already enough. To show $(9, 3)$-scatters aren't crystallized, the three scatters in Figure 4 suffice, just by considering angles around a vertex in the dual graph. Similarly, to check that $(16, 4)$-scatters aren't crystallized we only need the three periodic examples in Figure 5. The same kind of argument works for larger $(n, 3)$ and $(n, 4)$; all that's happening is that the angles at a vertex are too flexible for things to crystallize when n is large.

6 Crystallized scatters

For some choices of parameter there's a unique infinite scatter, namely $(3, 3)$, $(4, 4)$, $(8, 4)$, and $(6m, 6)$. So we're only concerned with four cases, when (n, k)=$(5, 3)$, $(7, 3)$, $(10, 4)$, and $(14, 4)$.

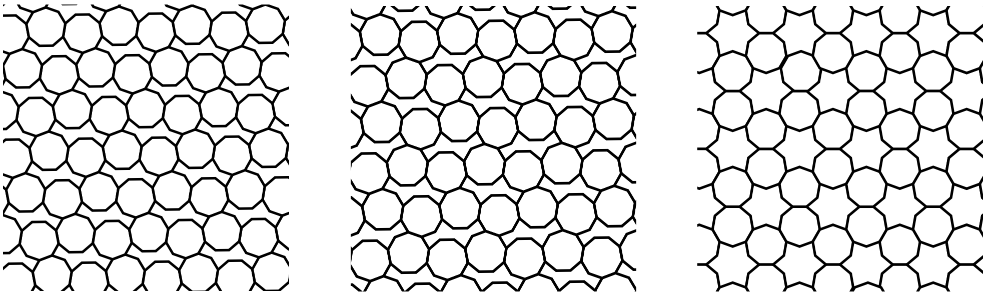

Figure 4: (9,3)-scatters.

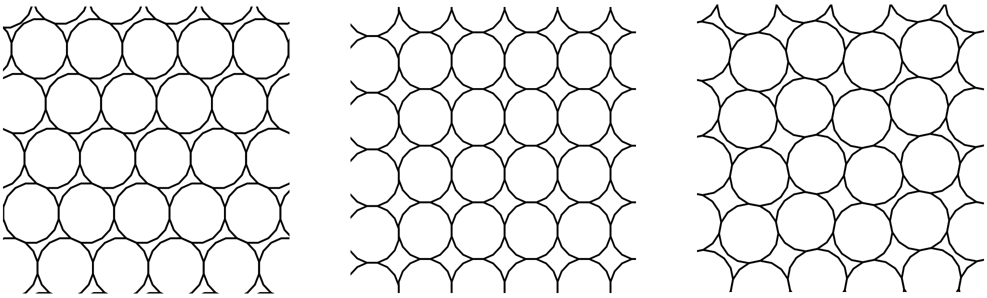

Figure 5: (16,4)-scatters.

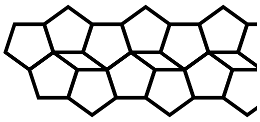

Figure 6: $(5,3)$-unavoidable.

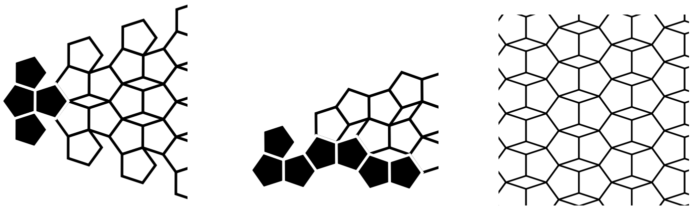

Figure 7: Proving that the shape in Figure 6 is unavoidable.

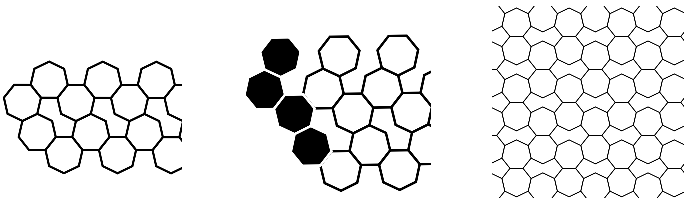

Figure 8: The middle and right shapes prove the shape on the left is unavoidable.

The unbounded shape P in Figure 6 is $(5,3)$-unavoidable, as follows. Let X be any $(5,3)$-scatter. If the shaded shape on the left in Figure 7 occurs anywhere, the infinite cone is forced by the minimal neighbor requirement, and the cone contains copies of P. If the shaded shape in the middle occurs, then the three row shape to the right is forced, forcing P again. Finally, suppose neither shaded shape occurs. Then one easily checks by exhaustion that X is isomorphic to the periodic scatter on the right, and P can be found along the diagonal.

Similarly, the unbounded shape on the left in Figure 8 is (7,3)-unavoidable. Any scatter avoiding the shaded initial tiles in the middle is equivalent to the periodic $(7,3)$-scatter on the right.

Finally, the unbounded shapes in Figure 9 and 10 are $(10,4)$- and $(14,4)$-unavoidable respectively. The proofs are as above. Any scatter avoiding the shaded initial tiles in the middle is equivalent to the periodic scatter on the right. In either case, the half infinite strip illustrated on the left is forced.

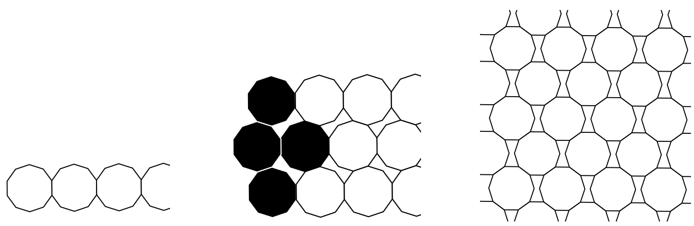

Figure 9: The shape on the left is $(10,4)$-unavoidable.

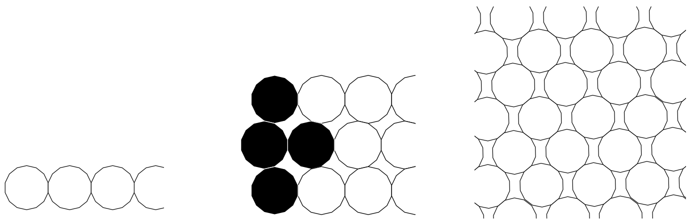

Figure 10: The shape on the left is (14, 4)-unavoidable.

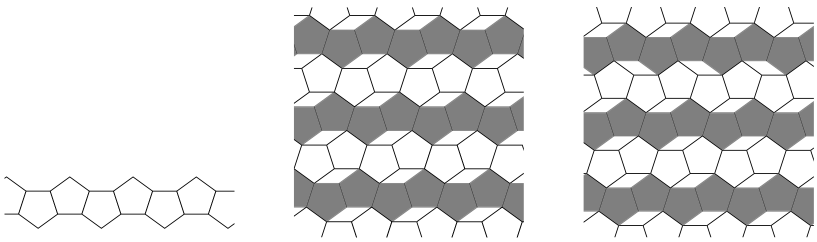

Figure 11: A (5, 2)-scatter and two (5, 3)-scatters made with translations.

7 Entropy

The *topological entropy* of the space of (n, k)-scatters X is defined as the exponential growth in complexity [Robs]:

$$h(X) = N \lim_{r\to\infty} \frac{1}{r^2} \log(c(r)),$$

where the *complexity* $c(r)$ is the number of different equivalent classes of a ball B_r of radius r, and N is a constant normalizing factor. We're only concerned here with whether the entropy is zero or positive, so the exact value of N isn't important.

Two tilings are equivalent if they agree after a small shift on a large disk. More precisely, we can take representatives of an equivalence class to have the first tile's position fixed and compute complexity by counting the number of ways to extend this single tile to an (n, k)-scatter intersected with a disk B_r.

There's uncountably many noncongruent $(5, 3)$-scatters just by gluing together copies of the infinite strip in Figure 11, but besides those built out of strips like this and two aperiodic examples (Figure 12) we don't know any others. So if we assume these are the only ones, then $\log(c(r)) = O(r)$ and $h(X) = 0$. Similar comments hold for $(n, k) = (7, 3)$, $(10, 4)$, and $(14, 4)$, where the only examples we know are periodic.

It's easy to prove that $h(X) > 0$ for $(4, 3)$-scatters with the following kind of argument. Divide a large $3n$ by $3n$ square into 3 by 3 squares. The center square of each of these can either be included in a $(4, 3)$-scatter or not, independently, since none of them have common neighbors, so $\log(c(r)) = \Omega(r^2)$ and $h(X) > 0$. Just by imitating the proof of

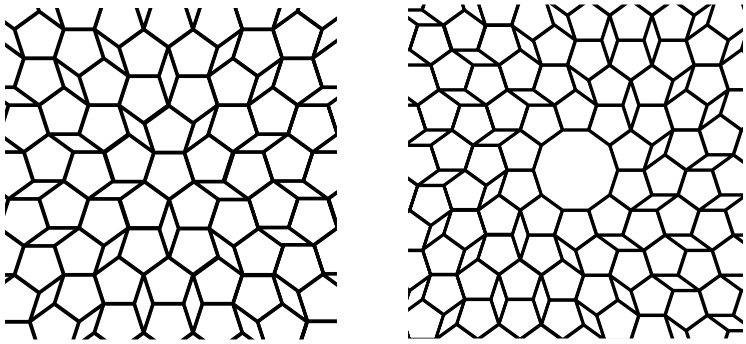

Figure 12: Crystallized scatters with finite symmetry groups.

Lemma 3.2 it seems true more generally that if $(n, k+1)$-scatters exist, then (n, k)-scatters exhibit positive entropy. It's still possible that $h(X) = 0$ for some non-crystallized cases, such as $(9, 3)$ and $(16, 4)$, but in any case we expect that crystallization implies zero entropy.

8 Open problems and generalizations

Several questions naturally arise. It's not clear what happens in the hyperbolic plane for example. Right now though, we don't know of any nontrivial cases of crystallization in this setting so it's not clear if this question is interesting.

Alternately, one might study scatters in higher dimensions with polytopal tiles. Deciding what happens with the dodecahedron and icosahedron in $\mathbb{R}^3$, for example, would be interesting since the pentagons are the most interesting examples in the two dimensional case. (For the ambitious, the 120-cell and 600-cell in $\mathbb{R}^4$.) Can questions like this be easily resolved with a computer program?

It seems crystallized scatters always exhibit nontrivial symmetry groups. At least every example we know so far does, but one should be careful since there are crystallized scatters with no translations, Figure 12. Both exhibit forbidden 5-fold rotational symmetry [Stei]. Are these the only aperiodic crystallized scatters? An example of the connection between rigidity in packing problems and symmetry is the following [Radi].

Theorem 8.1. *If there's only one optimally dense packing of Euclidean space, up to congruence, by congruent copies of bodies from some fixed, finite collection, then that packing must have a symmetry group with compact fundamental domain.*

Finally, is there always some finite set of maximal unavoidable shapes S (possibly unbounded themselves), meaning that a shape is unavoidable if and only if it's a subshape of some shape $s \in S$? From what we've computed here it's easy to check that this is true in the plane whether or not the scatters crystallize. But it might be nice to know this more generally as a suggestion that the set of unavoidable shapes is never too complicated.

9 Acknowledgements

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