Math 776 : Problem set 2

due: Thursday, March 15

$$\begin{aligned} (*) &= \text{required} \\ (**) &= \text{optional} \\ (***) &= \text{unsolved} \end{aligned}$$

1. (a) $(*)$ *Nonevasive* simplicial complexes are defined inductively as follows. If Δ consists of a single vertex, then Δ is nonevasive. If there exists a vertex $v \in \Delta$ such that $\mathrm{lk}_v(\Delta)$ and $\mathrm{dl}_v(\Delta)$ are both nonevasive, then Δ is nonevasive. Prove that nonevasive complexes are contractible.

   (b) $(**)$ Prove that this inductive definition of nonevasive is equivalent to the algorithmic definition we gave in class.

2. (a) $(*)$ If H is a graph, let $X(H)$ denote the flag complex of H. Prove that if every set of $2k+2$ vertices of H shares a neighbor, then $X(H)$ is k-connected.

   (b) $(**)$ Show that if every set of $2k+1$ vertices of H shares a neighbor, and the intersection of every $2k$ vertex stars in $X(H)$ is connected, then $X(H)$ is k-connected.

   (c) $(***)$ Prove or disprove that if every 3 vertices of H share a neighbor, and the intersection of every 2 vertex stars in $X(H)$ is connected, then $X(H)$ is 2-connected.

3. $(**)$ Let $d \geq 2$, and define a graph $G_{d,\epsilon}$ as follows. The vertices of $G_{d,\epsilon}$ are the unit sphere S^d, and two vertices are adjacent if the Euclidean distance between them is at least $2-\epsilon$. Show that for sufficiently small $\epsilon > 0$, we have that the chromatic number is given by $\chi\left(G_{d,\epsilon}\right) = d+2$.

4. (a) $(**)$ Show that for some constant $C > 0$ and all sufficiently large n there exist simplicial complexes $\Delta(n)$ on n vertices, with at least $Cn^{5/2}$ 2-dimensional faces, such that no subcomplex of $\Delta(n)$ is homeomorphic to a torus.

   (b) $(***)$ (Topological Turán theory.) Show that the exponent $5/2$ is best possible — that is, for every fixed $\epsilon > 0$ and sufficiently large n, every simplicial complex on n vertices with at least $n^{5/2+\epsilon}$ 2-dimensional faces contains an embedded torus. (This is a beautiful conjecture of Nati Linial. The corresponding facts with "torus" replaced by "sphere" were proved in Sós, Erdős, and Brown. "On the existence of triangulated spheres in 3-graphs, and related problems.")

5. $(**)$ Show that every 2-dimensional triangulated disk is collapsible. Give an example to show that the same is not true in general for d-dimensional disks.