MATH 776: GRAPH THEORY & COMBINATORICS II

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1. Tentative syllabus

We will consider "higher-dimensional" combinatorics in this course. The basic geometric objects of interest will be convex polytopes, the higher-dimensional analogues of the more familiar 3-dimensional polyhedra. We will follow selected chapters in Grünbuam's *Convex polytopes* and Ziegler's *Lectures on polytopes*.

1.0.1. *Geometric combinatorics.* Topics will likely include:

- in 3-dimensions — Steinitz's theorem
- in higher dimensions, the face lattice of a polytope, polarity
- classifying regular polytopes (the analogues of Platonic solids)?
- cyclic polytopes, simple and simplicial polytopes, recognizing a simple polytope by its 1-skeleton
- shellability, f-vector theory — upper bound theorem, a discussion of Stanley's proof of the "g-theorem" characterizing f-vectors of simplicial polytopes using algebraic geometric tools
- the Hirsch conjecture and Santos's recent 43-dimensional counterexample.
- Another type of geometric object of interest is hyperplane arrangements, and we may also discuss such combinatorial features as the lattice of flats and the number of chambers of a real hyperplane arrangement. Time permitting, we may also discuss the Goresky-MacPherson formula for homology of a subspace arrangement.
- A final topic in the "geometric" side of the course could be Guth and Katz's proof of the "joints conjecture" (a 3-dimensional Szemeredi-Trotter) and their subsequent resolution of the Erdos distinct distance problem in the plane.

1.0.2. *Topological combinatorics.* The basic topological objects of interest are posets, simplicial complexes, and more generally regular CW-complexes such as polyhedral complexes. Although these objects are fundamentally combinatorial in nature, we can study their topology via the "geometric realization" functor. We will draw on Anders Björner's chapter on "Topological methods" in the *Handbook of Combinatorics* (available on his webpage as a scanned-in .pdf) and Matousek's book *Using the Borsuk-Ulam theorem* as primary sources.

Topics will likely include:

- an introduction to simplicial (more generally, cellular) homology and cohomology, fundamental groups
- simplicial collapses and anti-collapses, Quillen's fiber lemmas, discrete Morse theory

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- shellable and constructible complexes
- Other special types of posets and complexes: Eulerian posets, geometric lattices, ...
- More f-vector theory — Kruskal-Katona theorem, the g-conjecture for simplicial spheres, the Charney-Davis conjectures.
- Quillen's conjecture on the lattice of elementary abelian p-subgroups of a group.
- Lovász's proof of the Kneser conjecture and "complexes of graph homomorphisms", other applications of the Borsuk-Ulam theorem to combinatorics.
- Complexes of all graphs with a monotone property and the evasiveness conjecture. Lattice of subgroups.
- Optional topics (somewhat depending on interests of the class) that we may touch on include: an introduction to oriented matroids and their representations, linear programming and combinatorial optimization, basic representation theory of the symmetric group.

2. Introduction to polytopes

We begin by studying convex polytopes. We will closely follow Ziegler's book [3] for this section of the course. The "classic" introduction to the subject is Grünbaum [2] and we will also occasionally look to this text for inspiration.

2.1. Definitions of polytopes.

We assume that everyone is comfortable with basic definitions such as *convex*, *convex hull*, *affine subspace*, and *affine hull*. These are reviewed in [3].

Following Ziegler, we have the following definitions.

Definition 2.1. *A $\mathcal{V}$-polytope is the convex hull of a finite set of points in some $\mathbb{R}^d$.*

An $\mathcal{H}$-polyhedron is the intersection of finitely many closed halfspaces in some $\mathbb{R}^d$. An $\mathcal{H}$-polytope is an $\mathcal{H}$-polyhedron that is bounded, in the sense that it does not contain a ray $\{\boldsymbol{x} + t\boldsymbol{y} : t \geq 0\}$.

The fundamental fact here (Theorem 1.1 in [3]) is that $\mathcal{V}$-polytopes and $\mathcal{H}$-polytopes are equivalent. This might be intuitively clear but it is not necessarily trivial to prove. (Indeed, this is most of the content of Chapter 1 in Ziegler's book.) From now on we will simply call them "polytopes."

For a moment, let's take this fundamental fact for granted and note the following consequences.

- Every intersection of a polytope with an affine subspace is a polytope.
- Every intersection of a polytope with a polyhedron is a polytope.
- The Minkowski sum of two polytopes is a polytope.
- Every projection of a polytope is a polytope.

Note that the first two are obvious for $\mathcal{H}$-polytopes and the last two for $\mathcal{V}$-polytopes.

Definition 2.2. *The dimension of a convex polytope is the dimension of its convex hull.*

Exercise 2.3. *Check that* 0*-dimensional polytopes are points and* 1*-dimensional polytopes are line segments, and that it follows that any two* 0*-polytopes or* 1*-polytopes are affinely equivalent. Give an example of two* 2*-polytopes which are not affinely equivalent.*

2.2. **Examples.** Let us list some of the fundamental examples of polytopes,

- convex polygons in $\mathbb{R}^2$
- the d-dimensional simplex, the convex hull of any d points not in an affine hyperplane
- d-dimensional cube $= \operatorname{conv}\{(\pm 1, \ldots, \pm 1)\}$
- d-dimensional cross-polytope $= \operatorname{conv}\{\pm \boldsymbol{e}_1, \pm \boldsymbol{e}_2, \ldots, \pm \boldsymbol{e}_n\}$
- the permutahedron $\Gamma_{d-1} \subset \mathbb{R}^d$
- the cyclic polytope $C_d(n)$

and also a few standard ways of building new polytopes from old ones.

- a prism over a polytope
- a cone or pyramid over a polytope
- a bipyramid over a polytope
- "stacking" a cone on one side of a polytope
- the product of two polytopes
- the polar dual of a polytope (more on this soon!)

2.3. **f-vectors of polytopes.** A *supporting hyperplane* of a polytope P is a hyperplane H which intersects P but does not disconnect P, i.e. a hyperplane such that $P \cap H \neq \emptyset$ and $P - H$ has only one connected component.

Definition 2.4. *A face of P is an intersection $P \cap H$ for some supporting hyperplane H.*

Note that a face of a polytope is itself a polytope. An alternate definition of face, sometimes more convenient, also includes the empty set and P itself as faces.

The f-vector of a polytope $(f_0, f_1, f_2, \ldots, f_d)$ records the number of i-dimensional faces f_i for every i.

Exercise 2.5. *Compute the f-vector for a d-dimensional simplex and for a d-dimensional cube.*

Exercise 2.6. *How many total faces does the d-dimensional simplex have? How many faces does the d-dimensional cube have?*

Exercise 2.7. *Give necessary and sufficient conditions for (f_0, f_1) to be the f-vector of a* 2*-polytope.*

Exercise 2.8. *Give necessary and sufficient conditions for (f_0, f_1, f_2) to be the f-vector of a* 3*-polytope.*

To give necessary and sufficient conditions for (f_0, f_1, f_2, f_3) to be the f-vector of a 4-polytope is an open problem. In fact even the following is apparently unknown [4].

Question 2.9. *(Eppstein, Kuperberg, Ziegler) Define the "fatness" of a* 4*-polytope P by*

$$\frac{f_1 + f_2}{f_0 + f_3}.$$

Is this fatness parameter bounded as P ranges over all 4-polytopes? Eppstein, Kuperberg, and Ziegler gave an example of a 4-polytope with fatness > 5.048 [1] but I am not sure if this is still the current record.

2.4. **Faces of cyclic polytopes.** Recall that the cyclic polytope $C_d(n)$ is the convex hull of n points on the moment curve in $\mathbb{R}^d$, where the moment curve is parameterized by

$$\boldsymbol{x}(t) = \left(t, t^2, \ldots, t^d\right).$$

We wish to describe the faces of $C_d(n)$. Let the n points be $\boldsymbol{x}(t_i)$ for $1 \leq i \leq n$ where

$$t_1 < t_2 < \cdots < t_d.$$

Then

$$\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ t_0 & t_1 & \cdots & t_d \\ \vdots & \vdots & \ddots & \vdots \\ t_0^d & t_1^d & \cdots & t_d^d \end{pmatrix} = \prod_{0 \leq i < j \leq d} (t_j - t_i)$$

(Vandermonde determinant). This gives immediately that no $d+1$ points on the moment curve are affinely dependent. Hence $C_d(n)$ is a *simplicial* polytope, i.e. every proper face is a simplex.

Now let $S = \{i_1, i_2, \ldots, i_d\} \subset [n]$. The question is how to tell whether S spans a facet in $C_d(n)$.

Theorem 2.10 (Gale's evenness criterion)**.** *The d vertices indexed by S span a facet if and only if the following evenness condition is satisfied: If $i < j$ are not in S then the number of $k \in S$ between i and j is even, i.e.*

$$2 \mid \#\{k \mid k \in S, i < k < j\}.$$

Proof. Define

$$F_S(\boldsymbol{x}) := \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \boldsymbol{x} & \boldsymbol{x}(t_{i_1}) & \cdots & \boldsymbol{x}(t_{i_d}) \end{pmatrix}.$$

We have that $F_S(\boldsymbol{x})$ is a linear function in $\boldsymbol{x}$, and $F_S(\boldsymbol{x}) = 0$ defines a hyperplane in $\mathbb{R}^d$ containing all the points indexed by S. Moreover, $F_S(\boldsymbol{x}(t))$ is a polynomial in t of degree d, and it has zeros at $t = t_{i_s}$ for $1 \leq s \leq d$.

Hence this polynomial has d distinct zeros and changes sign at each zero. The points indexed by S span a facet if and only if $F_S(\boldsymbol{x}(t_i))$ has the same sign for all the points $\boldsymbol{x}(t_i)$ with $i \in [n] \backslash S$. The desired result follows.

□

2.5. **Graphs of polytopes.** A few key theorems about graphs of polytopes, which we proved in class (following Chapter 3 of Ziegler's book):

Theorem 2.11 (Balinski's Theorem)**.** *The graph of a d-polytope is d-connected.*

Theorem 2.12 (Steinitz's Theorem)**.** *A graph G is the 1-skeleton of a 3-polytope if and only if G is 3-connected and planar.*

A famous conjecture about polytopes, now known to be false, is the Hirsch conjecture.

Conjecture 2.13 (Hirsch conjecture)**.** *If a d-dimensional polytope P has n facets, then the diameter δ of the graph of P is bounded by*

$$\delta \leq n - d.$$

Francisco Santos recently gave a counterexample to the Hirsch Conjecture with $d = 43$ and $n = 86$.

Question 2.14. *Is the Hirsch conjecture true for $n < 2d$?*

We proved in class a theorem of Naddef bounding the diameter 0-1 polytopes.

Theorem 2.15 (Naddef)**.** *If P is a d-dimensional* 0-1 *polytope then*

$$\delta \leq d,$$

and equality holds if and only if P is the d-dimensional cube.

Our final result on graphs of polytopes was that for a simple polytope P, the entire face lattice $L(P)$ can be recovered from its graph $G(P)$.

Theorem 2.16 (Blind and Mani)**.** *If P and P' are simple polytopes such that $G(P)$ is isomorphic to $G(P')$ (as graphs) then $L(P)$ is isomorphic to $L(P')$ (as posets).*

The theorem is due to Blind and Mani, and we followed Kalai's argument from, "A simple way to tell a simply polytope from its graph."

2.6. **Shelling and applications.** We define polytopal complexes, and shellability. We will see shortly that the boundary complex of every polytope is shellable — for contrast we mention that there are many examples of polytopal (indeed, simplicial) complexes which are homeomorphic to d-dimensional spheres but which are not shellable for every $d \geq 3$. We must conclude that some simplicial spheres are not realizable as the boundary complex of a simplicial polytope. In fact there are many more simplicial spheres than simplicial polytopes, but we will come back to this fact later.

The main theorems we are going for are the following.

First, the natural generalization of "$v - e + f = 2$" to higher dimensions.

Theorem 2.17 (Euler formula for d-dimensional polytopes)**.** *(To be filled in.)*

Next, a complete set of relations on f-vectors of simplicial polytopes.

Theorem 2.18 (Dehn–Sommerville relations for simplicial polytopes)**.** *(To be filled in.)*

The following theorem gives that cyclic polytopes have the maximal number of faces for a d-polytope with n vertices.

Theorem 2.19 (Upper bound theorem, McMullen)**.** *If P is a d-polytope with n vertices, then*

$$f_i(P) \leq f_i\left(C_d(n)\right)$$

for $0 \leq i \leq d$.

3. Introduction to hyperplane arrangements

For this section of the course, we followed Stanley's, "Introduction to Hyperplane Arrangements," which is available online as a .pdf file. (These notes were published in IAS / Park City: Geometric Combinatorics, edited by Miller, Reiner, and Sturmfuls.)

Over the course of a week, we covered parts of Lectures 1, 2, and 3. Some of the topics we covered:

- Möbius functions on posets and Möbius inversion
- Characteristic polynomials of hyperplane arrangements and counting regions of a real arrangement
- Chromatic polynomials of graphs, and the connection to characteristic polymomials
- Lattices of flats of matroids, and proving that intervals in the intersection poset of a hyperplane arrangement are geometric lattices

4. Topological combinatorics

4.1. **Borsuk–Ulam and applications.** In the first week, we covered the Borsuk–Ulam Theorem and some applications. We followed Matousek's book it Using the Borsuk–Ulam Theorem.

On the first day, we discussed several equivalent formulations of Borsuk–Ulam, used Borsuk–Ulam to prove the Brouwer fixed point theorem, and then gave a geometric proof of Borsuk–Ulam.

On the second day, we used Borsuk–Ulam to prove the Ham Sandwich theorem, and its discrete generalizations. We also proved the Polynomial Ham Sandwich Theorem. We then used the discrete Ham Sandwich theorem to prove two combinatorial theorems — one on rainbow convex sets and one on two thieves dividing a necklace with as few cuts as possible.

On the third day we proved Gale's theorem and saw haw Bárány used it to prove the Kneser conjecture. We briefly discussed Lovász's proof of the Kneser conjecture for contrast.

4.2. **Topological methods.** Now we are following Björner's "Topological Methods" chapter from the *Handbook of Combinatorics*.

Day one: gave a few Nerve Lemmas (without proof). Then proved the Crosscut Lemma, and used it to show that neighborhood complex of the Kneser graph $KG(n,k)$ is $(n-2k+1)$-connected, which completes the discussion of Lovász's proof of the Kneser conjecture. Then we defined chessboard complexes and used a Nerve Lemma argument to show that the chessboard complex $\Delta_{m,n}$ is $(\upsilon-2)$-connected, where

$$\upsilon = \min\{m, n, \lfloor \frac{1}{3}(m+n+1) \rfloor\}.$$

References

[1] David Eppstein, Greg Kuperberg, and Günter M. Ziegler. Fat 4-polytopes and fatter 3-spheres. In *Discrete geometry*, volume 253 of *Monogr. Textbooks Pure Appl. Math.*, pages 239–265. Dekker, New York, 2003.

[2] Branko Grünbaum. *Convex polytopes*, volume 221 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 2003. Prepared and with a preface by Volker Kaibel, Victor Klee and Günter M. Ziegler.

[3] Günter M. Ziegler. *Lectures on polytopes*, volume 152 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[4] Günter M. Ziegler. Convex polytopes: extremal constructions and f-vector shapes. In *Geometric combinatorics*, volume 13 of *IAS/Park City Math. Ser.*, pages 617–691. Amer. Math. Soc., Providence, RI, 2007.

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