# Math 776 : Problem set 1

due: Friday, March 2

$$\begin{aligned}(*) &= \text{required}\\ (**) &= \text{optional}\\ (***) &= \text{unsolved}\end{aligned}$$

1. $(*)$ For a $d$-dimensional polytope $P$, define its $f$-polynomial by
$$f_P(x) = f_0 + f_1 x + \cdots + f_d x^d.$$
(Since $P$ is $d$-dimensional, $f_d = 1$.) If $P$ and $Q$ are polytopes show that one can compute the $f$-polynomial of their product by
$$f_{P\times Q}(x) = f_P(x) f_Q(x).$$
Use this to find a formula for the number of $k$-dimensional faces of the $d$-dimensional cube.

2. $(*)$ We proved the upper-bound theorem for simplicial polytopes. Reduce the general case to the simplicial case by a small perturbation of the vertices. I.e. show that under a suitable perturbation of the vertices of a general polytope $P$ we get a simplicial polytope $\tilde{P}$ where $f_0(\tilde{P}) = f_0(P)$, and such that
$$f_i(\tilde{P}) \geq f_i(P)$$
for $i \geq 1$.

3. $(*)$ Show that the graph $G(P)$ of a $d$-dimensional polytope $P$ contains a subdivision of the complete graph $K_{d+1}$ as a subgraph. Conclude that if $d \geq 4$ then $G(P)$ is not planar.

4. $(**)$ Show that the number $f(d)$ of combinatorial equivalence classes of $d$-dimensional 0–1 polytopes satisfies the inequalities
$$2^{2^{d-2}} < f(d) < 2^{2^d}$$
for $d > 5$. (See Ziegler, Exercise 2.6 for a hint on the lower bound.)

5. $(**)$ Give the best lower and upper bounds you can for the number of combinatorial equivalence classes of $d$-dimensional polytopes with $n$ vertices.

6. $(**)$ The permutahedron $\Pi_{d-1}$ is the convex hull of all vectors obtained by permuting the coordinates of the vector $(12\ldots d)^T$. The vertices of $\Pi_{d-1}$ can be identified with the permutations in $S_d$ in a natural way, namely by associating $(x_1 x_2 \ldots x_d)^T$ with the vector that maps $x_i \to i$. Show that under this identification, two vertices are connected by an edge if and only if the corresponding permutations differ by an adjacent permutation.

7. $(***)$ Prove or disprove that for every $d$-dimensional polytope $P$ with $f$-vector $(f_0, f_1, \ldots, f_d)$,
$$f_i \geq \min(f_0, f_d),$$
for $0 \leq i \leq d$.

8. $(***)$ A $d$-polytope $P$ is called *dimensionally ambiguous* if there is another polytope $Q$ of dimension $d' \neq d$ with an isomorphic graph $G(Q) \cong G(P)$. Are the $d$-dimensional cubes dimensionally ambiguous? In particular, does there exist a $4$-polytope with graph isomorphic to the graph of the $5$-cube?

9. $(*)$ Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^n$ whose normals span a subspace $V$, and $\mathcal{B}$ another arrangement in $\mathbb{R}^n$, whose normals span a subspace $W$. Suppose that $V \cap W = \{0\}$. Then show that the characteristic polynomials of the arrangements $\mathcal{A} \cup \mathcal{B}$, $\mathcal{A}$, and $\mathcal{B}$ are related by
$$\chi_{\mathcal{A}\cup B}(t) = t^{-n}\chi_{\mathcal{A}}(t)\chi_{\mathcal{B}}(t).$$

10. $(*)$ Let $\mathcal{A}$ be the arrangement in $\mathbb{R}^n$ which is the union of the braid arrangement with the coordinate hyperplane arrangement. That is, the defining polynomial of $\mathcal{A}$ is
$$\prod_{1\leq i<j\leq n}(x_i - x_j)\prod_{1\leq i\leq n} x_i.$$
What is the number of regions $r(\mathcal{A})$?

11. $(*)$ Let $\mathcal{A}$ be the arrangement in $\mathbb{R}^n$ with equations $x_1 = x_2$, $x_2 = x_3$, $\ldots$, $x_n = x_1$. Compute the characteristic polynomial $\chi(\mathcal{A})(t)$, and the number of regions $r(\mathcal{A})$.

12. $(*)$
    (a) Give a formula with proof for $f(n, m)$, the number of regions in a hyperplane arrangement of $m$ hyperplanes in general position in $\mathbb{R}^n$.
    (b) Show that $f(n, m)$ is the maximum number of regions over all arrangements with $m$ hyperplanes in $\mathbb{R}^n$.

13. $(*)$ Prove that an interval in a geometric lattice is a geometric lattice.

14. $(**)$ Suppose that $G$ is a graph and $\mathcal{A}_G$ is the associated graphical arrangement. Show that if $G$ has a $k$-clique then $k! \mid r(\mathcal{A})$.