Expansion properties of random simplicial complexes

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Expander graphs

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They provide metric spaces that can only be embedded in Euclidean space with large distortion.
History

Existence of expanders first proved by Pinsker, using the probabilistic method.

Recent work of Breuillard, Green, and Tao.
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Edge expansion

The *Cheeger constant* $h$ of a graph $G$ is defined as follows.

$$h(G) = \min_{0 < |S| < |G|/2} \frac{|\text{e}(S, \bar{S})|}{|S|}.$$ 

Here $\text{e}(S, \bar{S})$ is the number of edges between a set of vertices $S$ and its complement $\bar{S}$, $|S|$ is the number of vertices in $S$, and the min is taken over all nonempty vertex subsets which are less than half the size of the graph.
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Define the *normalized Laplacian*, a linear operator on functions $f : V(G) \rightarrow \mathbb{R}$, by

$$\mathcal{L}[f] = f - A[f],$$

where $A[f]$ is the averaging operator

$$A[f](v) = \frac{1}{\deg(v)} \sum_{u \sim v} f(u).$$
Spectral expansion

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Let $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of the Laplacian $\mathcal{L}[G]$. Then the *spectral gap* of $G$ is the smallest positive eigenvalue $\lambda_2(G)$. 
A sequence of graphs \( \{G_i\} \) of bounded degree with \( |G_n| \to \infty \) is called an expander family if \( \lim_{n \to \infty} h(G_i) > 0 \).
Expander families

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Inequalities due to Cheeger and Buser in the continuous case, and Tanner, Alon, and Milman in the discrete case, relate \( h(G) \) and \( \lambda_2(G) \).
Inequalities

Let $\mathcal{L}$ be the *combinatorial Laplacian*, i.e.

$$\mathcal{L} = D - A,$$

where $D$ is the diagonal matrix with degrees along the diagonal and $A$ is the adjacency matrix. Let $\lambda_2$ be the smallest positive eigenvalue of $\mathcal{L}(G)$ for a connected graph $G$ with maximum degree $\Delta$ and Cheeger number $h$. 

Theorem

Cheeger / Buser inequalities

$$h^2 \Delta \leq \lambda_2 \leq 2h.$$
Inequalities

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\begin{theorem}
\textit{Cheeger / Buser inequalities}

$$
\frac{h^2}{2\Delta} \leq \lambda_2 \leq 2h
$$

\end{theorem}
Notions of higher-dimensional expanders

Linial and Meshulam’s use of “cohomological expansion” to find a sharp vanishing threshold for $H^1(Y, \mathbb{Z}_2)$, where $Y \in \mathcal{Y}(n, p)$ is a random 2-dimensional simplicial complex.

Gromov’s papers, “Topology, singularities, and expanders I & II” Fox, Gromov, Lafforgue, Naor, and Pach and “geometric overlap” properties of expanders, and Uli Wagner’s use of coarse expansion to give non-embeddable theorems.

Gromov and Guth’s recent proof that there exist isotopy classes of knots of arbitrarily large distortion, using expander-like properties of arithmetic hyperbolic manifolds.
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Foundational work of Garland on cohomology of buildings via “p-adic curvature”

Žuk’s work on property (T) in discrete groups and random groups

Ramanujan complexes — Li; Lubotzky, Samuels, and Vishne
Define $G(n, p)$ to be the probability space of graphs on vertex set $[n] = \{1, 2, \ldots, n\}$, where each edge has probability $p$, jointly independently.
Random graphs

Theorem (Erdős and Rényi, 1959)

Let $\epsilon > 0$ be fixed and $G \in G(n, p)$. Then

$$\Pr[G \text{ is connected}] \to \begin{cases} 1 & : p \geq (1 + \epsilon) \log n/n \\ 0 & : p \leq (1 - \epsilon) \log n/n \end{cases}$$
Define $Y(n, p)$ to be the probability space of 2-dimensional simplicial complexes with vertex set $[n]$, edge set $\binom{[n]}{2}$, and each 2-face has probability $p$. 
Random 2-complexes

**Theorem**

*(Linial–Meshulam, 2006)* Let $\epsilon > 0$ be fixed and $Y \in Y(n, p)$. Then

$$\Pr[H_1(Y, \mathbb{Z}/2) = 0] \rightarrow \begin{cases} 1 & : p \geq (2 + \epsilon) \log n/n \\ 0 & : p \leq (2 - \epsilon) \log n/n \end{cases}$$
Theorem

(Babson–Hoffman–K., 2011) Let $\epsilon > 0$ be fixed and $Y \in Y(n, p)$. Then

$$\Pr[\pi_1(Y) = 0] \rightarrow \begin{cases} 1 & : p \geq n^{-1/2} + \epsilon \\ 0 & : p \leq n^{-1/2} - \epsilon \end{cases}$$
Random 2-complexes — tools

The following topological lemma is useful in showing that $\pi_1$ is hyperbolic when $p \leq n^{-1/2-\epsilon}$.

Lemma

If $\Delta$ is a finite, connected 2-dimensional simplicial complex such that $f_2(\Sigma)/f_0(\Sigma) < 2$ for every subcomplex $\Sigma \subseteq \Delta$, then $\Delta$ is homotopy equivalent to a wedge of circles, spheres, and projective planes.
A group $G$ has Kazhdan’s property (T) if whenever it acts unitarily on a Hilbert space and has almost invariant vectors, it has a nonzero invariant vector.
Property (T)

A group $G$ has *Kazhdan’s property (T)* if whenever it acts unitarily on a Hilbert space and has almost invariant vectors, it has a nonzero invariant vector.

(More precisely, $G$ is (T) if the trivial representation of $G$ is an isolated point in its unitary dual equipped with the Fell topology.)
Random 2-complexes — results

Theorem

\[(\text{Hoffman–K.–Paquette, 2012})\text{ Let } \epsilon > 0 \text{ be fixed and } Y \in Y(n, p). \text{ Then}\]

\[
\Pr[\pi_1(Y) \text{ has property (T)}] \rightarrow \begin{cases} 
1 & : p \geq (2 + \epsilon) \log n/n \\
0 & : p \leq (2 - \epsilon) \log n/n
\end{cases}
\]
Theorem

(Żuk) If $\Delta$ is a finite, connected, pure 2-dimensional simplicial complex, such that for every vertex $v$, the link $lk_\Delta(v)$ is connected and has spectral gap of normalized Laplacian satisfying $\lambda_2[lk_\Delta(v)] > 1/2$, then $\pi_1(\Delta)$ has property ($T$).
Theorem

(Hoffman – K. – Paquette, 2012) Fix $k \geq 0$ and $\epsilon > 0$, and let $G \in G(n, p)$. Let $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq 2$ be the eigenvalues of the normalized Laplacian of $G$. There is a constant $C = C(k)$ so that when

$$p \geq \frac{(k + 1) \log n + C \sqrt{\log n \log \log n}}{n}$$

is satisfied, then

$$\lambda_2 > 1 - \epsilon,$$

with probability at least $1 - o(n^{-k})$. 
Let $G \in G(n, p)$ be an Erdős-Rényi random graph (on $n$ vertices with each edge having probability $p$, independently).
Random flag complexes

Let $G \in G(n, p)$ be an Erdős-Rényi random graph (on $n$ vertices with each edge having probability $p$, independently).

Let $X \in X(n, p)$ be the clique complex (or flag complex) of $G \in G(n, p)$, i.e. the maximal simplicial complex compatible with $G$. 
Random flag complexes — results

Theorem

(K., 2012) Let $0 < \epsilon < 1$ be fixed and $X \in X(n, p)$. Then

$$\Pr[H^k(Y, \mathbb{Q}) = 0] \rightarrow \begin{cases} 1 & : p \geq ((1 + k/2 + \epsilon) \log n/n)^{1/(k+1)} \\ 0 & : n^{-1/k} \ll p \leq ((1 + k/2 - \epsilon) \log n/n)^{1/(k+1)} \end{cases}$$

This provides a generalization of the Erdős–Rényi Theorem in a non-monotone setting.
Random flag complexes — tools

Theorem

(Garland, 1973, Ballman–Świątkowski, 1997) If $\Delta$ is a pure $k$-dimensional simplicial complex, such that the link $lk_{\Delta}(\sigma)$ of every $(k - 2)$-face $\sigma$ is connected and has spectral gap satisfying

$$\lambda_2[lk_{\Delta}(\sigma)] > 1 - 1/k,$$

then $H^{k-1}(\Delta, \mathbb{Q}) = 0$. 
Applying universal coefficients, and together with several earlier results on random flag complexes, we have the following.

**Corollary**

*Fix $d \geq 0$, and let $X \in X(n, p)$ be a random flag complex, where*

$$n^{-2/d} \ll p \ll n^{-2/(d+1)}.$$

*Then w.h.p. $X$ is $d$-dimensional and $\tilde{H}_i(X, \mathbb{Q}) = 0$ unless $i = \lfloor d/2 \rfloor$.*

I.e. almost all $d$-dimensional flag complexes have all their (reduced, rational) homology in middle degree.
A random flag complex $X(n, p)$ on $n = 100$ vertices with $0 \leq p \leq 0.6$. (Computation and image courtesy of Afra Zomorodian.)
Since the link of every face in a random flag complex is again a random flag complex, with a little more work we can show the following.

**Corollary**

Let $X \in X(n, p)$ be a random flag complex, where

$$n^{-2/d} \ll p \ll n^{-2/(d+1)}$$

then the $\lfloor d/2 \rfloor$-skeleton of $X$ is Cohen-Macaulay over $\mathbb{Q}$ w.h.p.
Open problems

How to handle torsion in homology of random complexes?
Conjecture

If $X \in X(n, p)$ is a random flag complex with

$$n^{-2/d} \ll p \ll n^{-2/(d+1)},$$

and if $d \geq 6$, then $X$ is homotopy equivalent to a wedge of $\left\lfloor d/2 \right\rfloor$-spheres.
Thanks for your time and attention.